ĐIỀU KIỆN CẦN TỐI ƯU CHO BÀI TOÁN TỐI ƯU HAI CẤP

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Tóm tắt

Bài toán tối ưu hai cấp đang hấp dẫn các nhà khoa học nghiên cứu do ý nghĩa khoa học và tính ứng dụng rộng rãi của bài toán trong thực tế. Tối ưu hai cấp xuất hiện trên sách báo, tạp chí thường có liên quan đến các hệ thống phân cấp. Bài toán tối ưu hai cấp bao gồm hai bài toán tối ưu, trong đó một phần dữ liệu của bài toán thứ nhất được xác định ẩn thông qua nghiệm của bài toán thứ hai. Người ra quyết định ở mỗi cấp cố gắng tối ưu hóa (cực tiểu hay cực đại) hàm mục tiêu riêng của cấp mình mà không để ý tới mục tiêu của cấp kia, nhưng quyết định của mỗi cấp lại ảnh hưởng tới giá trị mục tiêu của cả hai cấp và tới không gian quyết định nói chung. Mô hình toán học của bài toán tối ưu hai cấp cùng với công cụ dưới vi phân suy rộng dùng để thiết lập điều kiện tối ưu cho bài toán được chúng tôi trình bày trong bài báo này.

Từ khóa: *Bài toán, bài toán hai cấp, dưới vi phân suy rộng, nghiệm, tối ưu.*

NECESSARY CONDITIONS FOR BILEVEL OPTIMIZATION PROBLEM

Abstract

Bilevel optimzation is attracting scientists due to its scientific significance and wide applicability in practice. The bilevel programming in books and magazines is often related to hierarchies. The bilevel optimzation includes two optimal problems, in which a part of the data of the first problem is identified through the solution of the second problem. The decision maker at each level tries to optimize (minimum or maximum) the function of his own level without paying attention to the goal of the other level but the decision of each level affects the target value of both levels and the decision space in general. The math model of bilevel optimzation along with the convexificator tool used to establish optimal conditions for the problem is presented in this paper.

Keywords: *Problem, bilevel programming, convexificator, solution, optimization.*

JEL classification: C; C02

1. Introduction

Bilevel programming is arising from actual needs such as: Problem of socio-economic development planning for a territory or a country. Inside, the upper-level is the state that controls policies such as tariffs, exchange rate, import quota… with the aim of creating more jobs, minor resource usage… Lower-level are the companies with the goal of maximizing income with the constraints on superiors' policies. Or, resource allocation problem at a firm or a company with management decentralization. Inside, the upper echelon plays a central role in providing resources (capital, supplies and labor) aiming to maximize the company's profits. Lower-level are factories producing products in different locations, decide the ratio, own production output to maximize the performance of their units.

Mathetical model of bilevel programmimg problem (P) in this paper is a sequence of two optimization problems in which the feasible region of upper-level problem (P_1) is determined implicitly by the solutions set of the lower-level problem (P_2). It may be given as follows:

$$(P_1): \begin{cases} \text{Min} \quad F(x,y) \\ \text{subject to}: G(x,y) \le 0, y \in S(x); \end{cases}$$

Where, for each $x \in X$, $S(x)$ is the solution set of the following parametric optimization problem:

$$(P_2): \begin{cases} \text{Min} \quad f(x,y) \\ \text{subject to}: g(x,y) \le 0, \end{cases} \text{ where,}$$

$F=(F_1,...,F_m):\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^m,$

$f:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R},$

$G=(G_1,...,G_{m_2}):\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{m_2}$

and $g=(g_1,...,g_{m_1}):\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{m_1};n_i$ $m_i, i=1,2$ are integers, with $n_i\ge 1, m_i\ge 0.$

Whenever $m_1=0$ or $m_2=0$, this means that the corresponding inequality constraint is absen in the (P).

Within the scope of the article, we using convexificator for establishing necessary condition with optimal solution to the (P) in finite dimensional space.

2. Preliminaries

Let X be a Banach space, X^* topological dual of X, $\bar{x}\in X$. We recall some notions on convexificators in [4].

Definition 2.1 [4] The lower (upper) Dini directional derivatives of $f:X\to\overline{\mathbb{R}}\cup\{\pm\infty\}$ at $x\in X$ in a direction $\upsilon\in X$ is defined as

$$f_d^-(x;\upsilon)=\liminf_{t\downarrow 0}\frac{f(x+t\upsilon)-f(x)}{t}$$

$$\left(\text{resp., }\quad f_d^+(x;\upsilon)=\limsup_{t\downarrow 0}\frac{f(x+t\upsilon)-f(x)}{t}\right).$$

In case $f_d^-(x;\upsilon)=f_d^+(x;\upsilon)$ their common value is defined by $f'(x;\upsilon)$ which is called Dini derivative of f in the direction υ. The function f is called Dini differentiable at x its Dini derivative at x exists in all diretions.

Definition 2.2 ([4]) The function $f:X\to\overline{\mathbb{R}}\cup\{\pm\infty\}$ is said to have an upper (lower) convexificator $\partial^* f(x)$ at x if $\partial^* f(x)\subset X^*(\partial_* f(x)\subset X^*)$ is weakly* closed, and for all $\upsilon\in X$,

$$f_d^-(x,\upsilon)\le\sup_{x^*\in\partial^* f(x)}\langle x^*,\upsilon\rangle$$

$$\left(\text{resp., }\quad f_d^+(x,\upsilon)\ge\inf_{x^*\in\partial_* f(x)}\langle x^*,\upsilon\rangle\right)$$

A weakly* closed set $\partial^* f(x)\subset X^*$ is said to have a convexificator of f at x if it is both upper and lower convexificators of f at x.

Proposition 2.1 ([4]) Suppose that the function $f:X\to\mathbb{R}$ admits an upper convexificator $\partial^* f(\bar{x})$ at $\bar{x}\in X$. If f attains its minimum at $\bar{x}$ then: $0\in \text{clconv }\partial^* f(\bar{x})$ where cl and conv indicate the weakl* closure and convex hull, respectively.

Proposition 2.2 ([4]) Suppose that the function $f=(f_1,f_2,....f_n)$ be a continuous function from $\mathbb{R}^p$ to $\mathbb{R}^n$ and g be continuous function from $\mathbb{R}^n$ to $\mathbb{R}$. Suppose that, for each $i=1,...,n$, f_i admits a bounded convexificator $\partial^* f_i(\bar{x})$ and g admits a bounded convexifi-cator $\partial^* g(\bar{x})$ at $\bar{x}$. For each $i=1,...,n$ if $\partial^* f_i(\bar{x})$ and $\partial^* g(\bar{x})$ are upper semiconti-nuos at $\bar{x}$ then the set:

$$\partial^*(f\circ g)(\bar{x})=\partial^* g(f(\bar{x}))(\partial^* f_1(\bar{x}),...,\partial^* f_n(\bar{x}))$$

is a convexificator of $f\circ g$ at $\bar{x}$.

We shall begin with establishing necessary optimality condition for optimal solutions of bilevel programming problem.

3. Optimality condition

A pair $(\bar{x},\bar{y})$ is said to be optimal solution to the (P) if it is an optimal solution to the following problem: $\text{Min}_{(\bar{x},\bar{y})\in\bar{S}}F(x,y)$ where,

$$\bar{S}=\left\{(\bar{x},\bar{y})\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}:G(x,y)\le 0;\ y\in S(x)\right\}.$$

According to Stephane Dempe [3], (P) can be replaced by (P^*): Min $F(x,y)$,

subject to:

$$G(x,y)\le 0,\ g(x,y)\le 0;\ f(x,y)-V(x)\le 0$$

với $(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$.

Luderer (1983) has proved that the problem (P^*) has an optimal solutions, where

$$V(x)=\min_y\left\{f(x,y):g(x,y)\le 0, y\in\mathbb{R}^{n_2}\right\}.$$

Assumption 3.1 Dempe [2] has proved that under the following hypotheses $(H_1),(H_2),(H_3)$, the problem (P) has at lest one optimal solution.

(H_1): $F(.,.), f(.,.), g(.,.)$ and $G(.,.)$ are lower semicontinuos (l.s.c) on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$;

(H_2): $V(.,.)$ is upper semicontinuos (u.s.c) on $\mathbb{R}^{n_1}$;

(H_3): The problem (P^*) has at least one feasible solution, there exists $\upsilon^* < c < \infty$ such as:

$$M = \{(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2} : G(x,y)\le 0, \quad g(x,y)\le 0, F(x,y)\le c\}$$

is not empty and bounded.

Definition 3.1 [1] The problem (P) is said to be regular at $(\bar{x},\bar{y})$ if there exists a neighborhood U of $(\bar{x},\bar{y})$ and $\delta,\beta>0$ such that

$$\forall(\mu,\upsilon)\in\mathbb{R}_+^{m_1}\times\mathbb{R}_+^{m_1};\forall(x,y)\in U;$$
$$\forall x_g^*\in \text{co } g(x,y);\forall x_G^*\in\text{co } G(x,y);$$
$$\forall x_f^*\in\text{co } f(x,y);\forall x_V^*\in\partial^* V(x)\times\{0\};$$
$$\exists\xi\in\delta\mathbb{R}_X$$

such that

$$\mu g(x,y)+\upsilon G(x,y)+\left\langle (x_g^*, x_G^* x_f^* - x_V^*, \xi)\right\rangle \ge 0.$$

Theorem 3.1 Let $(\bar{x},\bar{y})$ is a solution of (P). Suppose that, there exists a neigh-borhood $U\subset X$ at $(\bar{x},\bar{y})$ such that the functions F, f, g, G are continuous on U and admits bounded convexificators.

$$\partial^* F(\bar{x},\bar{y});\partial^* f(\bar{x},\bar{y});\partial^* g(\bar{x},\bar{y});\partial^* G(\bar{x},\bar{y})$$

at $(\bar{x},\bar{y})$.

Assume that:

$$\partial^* F;\partial^* f;$$
$$\partial^* g;\partial^* G$$

are upper semicontinuos at $(\bar{x},\bar{y})$.

Then, there exists scalars $\lambda_1,\lambda_2,...,\lambda_{m+1}$ $\gamma\ge 0$ and vector

$$\mu=\left(\mu_1,...,\mu_{m_1}\right)\in\mathbb{R}_+^{m_1};$$

$\upsilon=\left(\upsilon_1,...,\upsilon_{m_2}\right)\in\mathbb{R}_+^{m_2}$ such that:

$$\begin{cases} \left\|(\mu,\upsilon,\gamma)\right\|=1 \quad and \quad \sum_{k=1}^{m}\lambda_k=1; \\ \sum_{i=1}^{m_1}\mu_i g_i(\bar{x},\bar{y})=0 \quad and \quad \sum_{j=1}^{m_2}\mu_j G_j(\bar{x},\bar{y})=0; \\ (0,0)\in cl\{\sum_{k=1}^{m}\lambda_k conv\,\partial^* F_k(\bar{x},\bar{y})+ \\ \lambda_{m+1}[\gamma conv\,\partial^* f_k(\bar{x},\bar{y})+\sum_{i=1}^{m_1}\mu_i conv\,\partial^* g_i(\bar{x},\bar{y}) \\ +\sum_{j=1}^{m_2}\upsilon_j conv\,\partial^* G_j(\bar{x},\bar{y})-\gamma(\partial^* V(\bar{x})\times\{0\}]\}, \ (1) \end{cases}$$

where,

$$\begin{cases} \partial^* V(\bar{x}) = conv\ \{\partial^* f(.,y)(\bar{x}) : y\in J(\bar{x})\} \\ J(\bar{x})=\{y\in\mathbb{R}^{n_2} : g(\bar{x},y)\le 0; f(\bar{x},y)=V(\bar{x})\}. \end{cases}$$

If in addition to the above assumptions, the problem (P) is regular at $(\bar{x},\bar{y})$ one has $\lambda_k>0,\ k=\overline{1,m}.$

Proof

A ccording to Stephane Dempe [3], (P) can be replaced by (P^*): Min $F(x,y)$, subject to:

$$G(x,y)\le 0,\ g(x,y)\le 0;\ f(x,y)-V(x)\le 0$$

with $\forall(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$.

Applying the scalarization theorem by Gong (2010, Theorem 3.1) to Problem (P^*) yields the existence of a continuous positively homogeneous subadditive function Λ on $\mathbb{R}^m$ satisfying

$$y_2-y_1\in\text{int}\mathbb{R}_+^m\Rightarrow\Lambda(y_2)<\Lambda(y_1)$$

and : $(\Lambda\circ F)(x,y)\ge 0,\forall(x,y)\in M$

Hence, $(\bar{x},\bar{y})$ is a minimum of the following scalars optimization Problem (MP): Min $(\Lambda\circ F)(x,y)$,

subject to:

$$G(x,y)\le 0,\ g(x,y)\le 0;\ f(x,y)-V(x)\le 0$$

with $\forall(x,y)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$.

Luderer (1983) has proved that the problem (MP) has an optimal solutions, where

$$V(x)=\min_y\left\{f(x,y): g(x,y)\le 0, y\in\mathbb{R}^{n_2}\right\}.$$

Taking account of Theorem 1 in Bard

(1984) to the scalars problem (MP) yields the exists $\theta,\ \theta_1,\ \gamma \geq 0$ and vector $\mu = \left(\mu_1,...,\mu_{m_1}\right) \in \mathbb{R}_+^{m_1};\ \upsilon = \left(\upsilon_1,...,\upsilon_{m_2}\right) \in \mathbb{R}_+^{m_2}$ such that

$$\begin{cases} \left\|(\mu,\upsilon,\gamma)\right\| = 1 \text{ and } \theta + \theta_1 = 1 \\ \sum_{i=1}^{m_1} \mu_i g_i(\bar{x},\bar{y}) = 0 \text{ and } \sum_{j=1}^{m_2} \mu_j G_j(\bar{x},\bar{y}) = 0 \\ (0,0) \in cl\{\theta \text{conv } \partial^*(\Lambda \circ F)(\bar{x},\bar{y}) \\ +\theta_1[\sum_{i=1}^{m_1} \mu_i \text{conv } \partial^* g_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \upsilon_j \text{conv } \partial^* G_j(\bar{x},\bar{y}) \\ +\gamma \text{conv } \partial^* f(\bar{x},\bar{y}) - \gamma(\partial^* V(\bar{x}) \times \{0\})]\} \qquad (2) \end{cases}$$

where,

$$\begin{cases} \partial^* V(\bar{x}) = \text{conv } \{\partial^* f(.,y)(\bar{x}) : y \in J(\bar{x})\} \\ J(\bar{x}) = \{y \in \mathbb{R}^{n_2} : g(\bar{x},y) \leq 0;\ f(\bar{x},y) = V(\bar{x})\}. \end{cases}$$

We now check hypotheses of a chain rule by Jeyakumar-Luc ([4], Prop 5.1) to the composite function $(\Lambda \circ F)(x,y)$. Since the function Λ is continuous convex, we can apply Proposition 2.2.6 trong Clarke (1983) to deduce that it is locally Lipschitz.Hence, its subdifferential $\partial_C \Lambda(F(x,y))$ is abounded convexificator of Λ at $F(\bar{x},\bar{y}) = 0$. Since function Λ is convex and locally Lipschitz, Proposition 7.3.9 trong Schirotzek (2007) was poined that:

$$\partial_C \Lambda(F_{(\bar{x},\bar{y})}(\bar{x},\bar{y})) = \partial \Lambda(F_{(\bar{x},\bar{y})}(\bar{x},\bar{y})).$$

Moreover, due to Proposition 2.2, we have $\partial_C \Lambda(F_{(\bar{x},\bar{y})}(\bar{x},\bar{y}))(\partial^* F(\bar{x},\bar{y})_1,...,\partial^* F(\bar{x},\bar{y})_m)$ is a convexificator of $(\Lambda \circ F)(\bar{x},\bar{y})$ at $(\bar{x},\bar{y})$.

It follows from (2), we obtain

$$(0,0) \in \text{cl}\{\theta\, \partial_C \Lambda(F_{(\bar{x},\bar{y})}(\bar{x},\bar{y}))(\partial^* F(\bar{x},\bar{y})_1,..., \partial^* F(\bar{x},\bar{y})_m) + \theta_1[\sum_{i=1}^{m_1} \mu_i \text{conv } \partial^* g_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \upsilon_j \text{conv } \partial^* G_j(\bar{x},\bar{y}) + \gamma \text{conv } \partial^* f(\bar{x},\bar{y}) - \gamma(\partial^* V(\bar{x}) \times \{0\})]\}. \qquad (3)$$

Since $F_{(\bar{x},\bar{y})}(\bar{x},\bar{y}) = 0$, it follows from (3) that there exists a sequence

$$z_n \in \text{cl}\{\theta\, \partial_C \Lambda(0)(\partial^* F(\bar{x},\bar{y})_1,...,\partial^* F(\bar{x},\bar{y})_m) + \theta_1[\sum_{i=1}^{m_1} \mu_i \text{conv } \partial^* g_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \upsilon_j \text{conv } \partial^* G_j(\bar{x},\bar{y}) + \gamma \text{conv } \partial^* f(\bar{x},\bar{y}) - \gamma(\partial^* V(\bar{x}) \times \{0\})]\}, \qquad (4)$$

such that $\lim_{n\to+\infty} z_n = 0$. By (4), there exists a sequence $\{\chi_n\} \subset \partial_C \Lambda(0) \subset \mathbb{R}^m$ such as

$$z_n \in \text{cl}\{\theta\, \chi_n(\partial^* F(\bar{x},\bar{y})_1,...,\partial^* F(\bar{x},\bar{y})_m) + \theta_1[\sum_{i=1}^{m_1} \mu_i \text{conv } \partial^* g_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \upsilon_j \text{conv } \partial^* G_j(\bar{x},\bar{y}) + \gamma \text{conv } \partial^* f(\bar{x},\bar{y}) - \gamma(\partial^* V(\bar{x}) \times \{0\})]\}, \qquad (5)$$

Since $\partial_C \Lambda(0)$ is a compact set in $\mathbb{R}^m$, we can assume that $\chi_n \to \chi \in \partial_C \Lambda(0)$. Putting $\lambda = \theta.\chi$ one has $\lambda = (\lambda_1,...,\lambda_m).\ \in \mathbb{R}^m$. It follows from (5), we have

$$(0,0) \in \text{cl}\{\lambda(\partial^* F(\bar{x},\bar{y})_1,...,\partial^* F(\bar{x},\bar{y})_m) + \theta_1[\sum_{i=1}^{m_1} \mu_i \text{conv } \partial^* g_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \upsilon_j \text{conv } \partial^* G_j(\bar{x},\bar{y}) + \gamma \text{conv } \partial^* f(\bar{x},\bar{y}) - \gamma(\partial^* V(\bar{x}) \times \{0\})]\}, \qquad (6)$$

Sine $\lambda = (\lambda_1,...,\lambda_m)$, putting $\theta_1 = \lambda_{m+1}$, by virtue of (6), its holds that

$$(0,0) \in \text{cl}\{\sum_{k=1}^{m} \lambda_k \text{conv } \partial^* F_k(\bar{x},\bar{y}) + \lambda_{m+1}[\sum_{i=1}^{m_1} \mu_i \text{conv } \partial^* g_i(\bar{x},\bar{y}) + \sum_{j=1}^{m_2} \upsilon_j \text{conv } \partial^* G_j(\bar{x},\bar{y}) + \gamma \text{conv } \partial^* f(\bar{x},\bar{y}) - \gamma(\partial^* V(\bar{x}) \times \{0\})]\},$$

which implies (6).

Let us see that $\lambda_i \in \mathbb{R}_+^m \setminus \{0\} (\forall i = 1,...,m)$. Since $\theta \geq 0,$ we just need to prove $\chi \in \mathbb{R}_+^m \setminus \{0\}$. Indeed, for any it can be written $0 - (-y) \in \text{int } \mathbb{R}_+^m$. We have

$$\begin{aligned} \left\langle \chi, -y \right\rangle &= \left\langle \chi, F_{(\bar{x},\bar{y})}(\bar{x},\bar{y}) - y - F_{(\bar{x},\bar{y})}(\bar{x},\bar{y}) \right\rangle \\ &\leq \Lambda(F_{(\bar{x},\bar{y})}(\bar{x},\bar{y}) - y) - \Lambda(F_{(\bar{x},\bar{y})}(\bar{x},\bar{y})) \\ &= \Lambda(-y) < P(0) = 0. \end{aligned}$$

Consequently, $\chi \in \mathbb{R}_+^m \setminus \{0\},$ and so, it can be assumed that $\sum_{k=1}^{m} \chi_k = 1$. Hence, $\sum_{k=1}^{m+1} \lambda_k = 1$, which completes the proof.

4. Conclusions

In reference [2], the author applies bilevel programming problem with the function considered is scalar function in $\mathbb{R}$.

In our article, the optimal condition of the problem is established for function vector in $\mathbb{R}^m$. This is a new result, have scientific meaning and tools to prove the problem is convexificator, currently many scientists are interested.

TÀI LIỆU THAM KHẢO

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